Falkner-Skan approximation for gradually variable flows

M. Argentina

Departamento de Física, Universidad de Chile, Casilla 487-3, Santiago, Chile.

E. Cerda

Departamento de Física de la Universidad de Santiago de Chile, Avda Ecuador 3493, Casilla 307 Correo 2, Santiago, Chile

Present address: DAMTP, Silver Street, CB3 9EW Cambridge, United Kingdom.

Abstract

We discuss here a method for computation of gradually variable laminar flows for large Reynold number. The model consists in approximating locally the flow with self similar profiles. This approach permits a derivation of two coupled ordinary differential equations. One of them is the Falkner-Skan equation with specific boundary conditions that once solved permits to study variable flows in quite different problems or geometries. We apply the model to the problem of the Poiseuille flow, and compare it with the solution obtained by integrating directly the fluid motion equation.

Key words: Fluid mechanics, Boundary layer, Falkner-Skan, Self-similar, Poiseuille flow, inviscid-viscous flow transition.

1 Introduction

When an icompressible fluid passes in the vicinity of solid boundaries, the Navier Stokes equations may be reduced drastically. In order to introduce the boundary layer theory, Prandtl [1] proposed that viscous effects would be confined to a thin shear layer adjacent to the boundaries in the case of a motion with very little viscosity, i.e. in the case of flows for which the characteristic Reynold number, Re, is large. Hence, the fluid is split into two parts: near the boundaries the viscosity effect are important and the fluid is said to be *viscous*, and far away form the boundaries, where viscous forces are unimportant and the fluid is said to be *invicid*. The *inviscid* fluid induces a gradient of pressure in the boundary layer, and in some sense *leads* the viscous fluid. When the fluid is moving along a solid obstacle, it looses velocity in the neighborhood of the surface body, and the *viscous* part of the fluid tends to invade the *inviscid* region. In the present article, we study how the transition form a perfect fluid to a viscous one is performed.

The Boundary Layer Theory (BLT) theory explain qualitatively and quantitatively the famous Blasius flow [2], i.e. the steady flow over a flat plate at zero incidence. A comparison between the Blasius solution and Wortmann's visualization of the flow [3] demonstrates that BLT work in this case [4], until very large value of the Reynold number. Another example is those of a solid cylinder moving at constant velocity in a fluid. In order to apply the (BLT), the *inviscid* fluid is supposed to have the velocity field of perfect fluid in contact with the cylinder. With such an assumption, the external velocity is $U = U_0 \sin\theta$. The variable θ measures the angle from the stagnation point. Terrill showed numerically [5] that the solutions obtained using the BLT, with this asymptotic velocity terminates in a Goldstein singularity [6] at $\theta = \theta_c \sim 104.5^o$. The singularity occurs at the point where a reversed flow is about to develop. This a serious problem because solutions cease to exist after the separation point θ_c. For this Reynold number limit, the point of separation is observed experimentally at $\theta \sim 70^o$ [7], and the BLT does not capture quantitatively the physical properties. In his thesis, Hiemenz [7] reported measurement of the pressure around the cylinder and showed a discrepency with the pressure obtained with the perfect fluid approximation. These experimental datas, associated with the (BLT) permitted a prediction of the separation point at $\theta \sim 70^o$. Hence, the BLT seems to be a good approximation, if the pressure imposed to the boundary layer is correctly evaluated.

In this article, we propose a method that permits to couple the pressure to the boundary layer equation. We approximate locally the velocity profiles with self similar ones. In order to explain how this modelisation works, we solve the flow of an icompressible fluid contained between two parallel flat planes. We give at entrance of the pipe a flat profile for the horizontal velocity, and study how the system relaxes, far away from this point, to its asymptotic regime, i.e. the Hagen-Poiseuille solution.

2 Classical Boundary Layer Theory

2.1 Formulation

We consider a flow of an incompressible fluid with density ρ and a dynamic viscosity $\mu = \nu\rho$, past a body with a typical length L_0. This is described with the Navier-Stokes equations that take the form

$$\partial_t \overrightarrow{u} + \left(\overrightarrow{u}.\overrightarrow{\nabla}\right)\overrightarrow{u} = -\frac{1}{\rho}\overrightarrow{\nabla}P + \nu\overrightarrow{\nabla}^2\overrightarrow{u} \quad (1a)$$

$$\overrightarrow{\nabla}.\overrightarrow{u} = 0, \quad (1b)$$

with no slip boundary conditions for the velocity along the body surface, i.e., the normal and the tangential component of the velocity are zero. The transition from zero velocity at the wall to the full magnitude at some distance from it, takes place in a very thin layer, the so called boundary layer (whose thickness is defined by δ_0), where the velocity gradients normal to the walls are very large. If we assume that the horizontal velocity (parallel to the surface's body) scales like U_0, and the typical distance along the solid body is L_0, a dimensional analysis gives the behavior of δ_0 as function of the physical parameters:

$$\delta_0 \sim \sqrt{\frac{\nu L_0}{U_0}}$$

This reports that the depth of the boundary layer increases when the velocity U_0 decreases, and increases when the typical distance along the body L_0 increases. An another way to understand the latter property is the following: since near the body the viscosity forces tend to diminish the velocity inside the boundary layer, thus its thickness must increase by mass conservation.

We now reduce the Eqs. (1) to the Prandtl model [1]. BLT applies to flows where there exists a thin shear layer, say of typical width $\delta_0 \ll L_0$. We note u (resp. v) as the tangential (resp. normal) velocity with respect to the body surface. We use the scalings $x \sim L_0$, $y \sim \delta_0$, $u \sim U_0$ and $v \sim V_0$. The relation (1b) gives $V_0 \sim \frac{\delta_0}{L_0}U_0$: the vertical velocity is small compared to the horizontal one. Using these scaling the Navier-Stokes equations are reduced to the Prandtl equation:

$$\partial_t u + uu_x + vu_y = -P_x + Re^{-1}\left(u_{yy}\right) \quad (2a)$$

$$0 = -P_y \quad (2b)$$

$$u_x + v_y = 0 \quad (2c)$$

Hence, the pressure (redefined such that $P \rightarrow \rho P$) only varies in the direction normal to the boundary layer. As usual, the Reynold number is defined by

$$Re = \frac{U_0 L_0}{\nu}$$

For derivation of the Prandtl equations, it has been supposed that the Reynold number is high enough (typically, $Re \sim \frac{{L_0}^2}{\delta_0^2}$). For a flow past rigid body, the boundary conditions are

$$u|_{y=0} = v|_{y=0} = 0 \tag{3a}$$
$$u|_{y\to\infty} = U(x,t) \tag{3b}$$

where $U(x,t)$ is the inviscid velocity far away from the body, and obeys to the Euler equation. A way to compute the pressure $P(x)$ consists in applying the Bernouilli relation in the inviscid region (where the velocity is also supposed to be irrotational), and we deduce

$$P_x = -UU_x$$

Hence, in the Prandtl modelisation, either the pressure P or the inviscid velocity must be given. We are interested here to stationary flows, and form now on, we will asssume that $\partial_t u = 0$.

2.2 Self similar solution and Falkner-Skan equations

There is no general solution known for the Prandtl equation (2) with boundary conditions (3). Nevertheless, Blasius [2] has found a self-similar solution to this problem when $U(x)$ is constant. This models the flow of a fluid over a plate at zero incidence, and the agreement with experiment is excellent (at least for moderately large Reynold number). Later, Falkner and Skan [9] has found families of self-similar solutions if $U(x)$ is proportional to a power of x, a fact that we will comment in the following paragraphs.

A general transformation inspired by Meksyn [10] and formalized by Görtler [8], permits to change Eq. (2) into appropriate (self-similar) coordinates. In the case of cartesian geometry, this transformation is

$$\xi = \int_0^x U(x)dx \tag{4a}$$
$$\eta = \frac{Uy}{\delta} \tag{4b}$$
$$\delta = \sqrt{\frac{2\xi}{Re}} \tag{4c}$$
$$\psi = \delta f(\xi,\eta) \tag{4d}$$

where ψ is the usual current function, i.e. $u = \psi_y$ and $v = -\psi_x$. The boundary layer thickness, δ_0, is proportional to δ. The physical domain, the plane (x, y) is mapped into (ξ, η). The unknown f is understood as a (self-similar) stream function. With such a transformation, the Prandtl's equation is transformed into the Görtler equation:

$$f''' + ff'' + \beta(1 - f'^2) = 2\xi \left(f' f'_\xi - f_\xi f''\right), \tag{5}$$

The prime refers to derivative with respect to η. The new variable β, defined by Görtler as the principal function, is

$$\beta = 2\xi \frac{U_x}{U^2} \tag{6}$$

Self-similar solutions exist only when f and f' does not depend on ξ. In such a case, β must be constant and, according to the Görtler transformation (4a), we deduce:

$$\frac{dU}{dx} = U(\xi)\frac{dU}{d\xi} = \frac{\beta U^2}{2\xi}$$

this permit to find that $U \sim \xi^{\frac{\beta}{2}}$. Since we have $\xi_x = U$, we deduce finally that

$$U(x) = U_0(x - x_0)^{\frac{\beta}{2-\beta}}. \tag{7}$$

With such potential velocity, self-similar solutions obey to the Falkner-Skan equation [9]

$$f''' + ff'' + \beta(1 - f'^2) = 0. \tag{8}$$

The boundary conditions (3) mapped into the (ξ, η) plane are

$$f(0) = 0 \tag{9a}$$

$$f'(0) = 0 \tag{9b}$$

$$\lim_{\eta\to\infty} f'(\eta) = 1 \tag{9c}$$

Potential flows like those defined in (7) occur, in fact, in the neighborhood of the stagnation point of a wedge with an angle $\beta\pi$, and it can be deduced from transformal conformation techniques [11].

Since the system (8) is a non-linear third dimensional ordinary differential equation, and due to the boundaries (9), numerical solutions are obtained with a shooting method, taking as a free parameter $f''(0)$. This term is proportional to the shear of the fluid at the boundary since

$$\begin{cases} u = Uf'[\eta] \\ u_y = \frac{U^2}{\delta} f''[\eta] \\ u_{yy} = \frac{U^3}{\delta^2} f'''[\eta] \end{cases}.$$

In the above relations, it is seen the relation between the curvature of the velocity profile and $f'''(\eta)$. Note that along the body boundary, we have $f'''(0) = -\beta$, that is the usual relation between the pressure and the curvature at the surface of the body: $u_{yy}(x, 0) = -P_x \equiv UU_x$.

The solutions of the Falkner-Skan equation has been studied numerically by Hartree [12]. In the case of accelerating flows ($\beta > 0$), the velocity profiles have no points of inflection, whereas in the case of decelerated flows ($\beta < 0$),

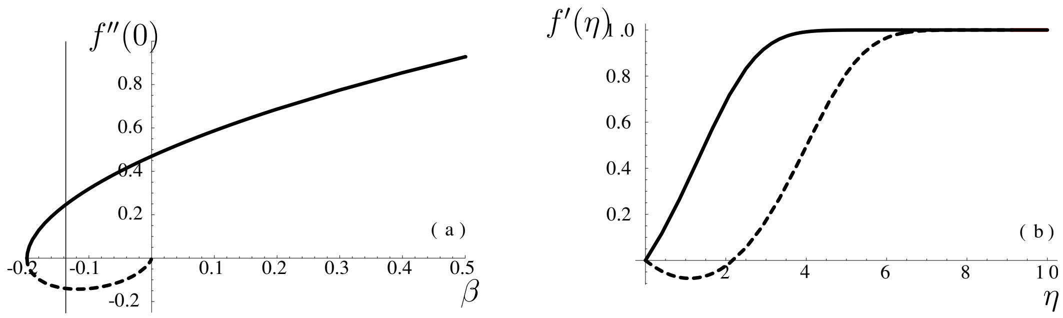

Fig. 1. (a) $f''(0)$ as function of β . The solid line represents profile branch with no reverse flow. The dashed line represents profile branch with reverse flow; (b) Two profiles obtained with $\beta = -0.13$. The solid line represents the profile of the solid branch of Fig (a). The dashed line represents the profile of the dashed branch of Fig (b) and exhibits a reverse flow region.

they exhibit a point of inflection (as seen in Fig. 1.b). At $\beta = \beta^* = -0.199$, separation occurs, and the profile has a region of reverse flow ($f''(0) < 0$) as shown in Fig. 1.b. For smaller values of β, no more solution respecting the boundary conditions (9) are found and Stewartson [13] subsequently demonstrated that for $\beta < \beta^*$, all solutions have the properties $f'(\infty) > 1$. The existence of two possible profiles for negative values of β (see Fig. 1.a) may be at the origin of the Goldstein singularity, that occurs when a region with a reverse-flow ($f''(0) < 0$) is about to appear [14].

3 The locally self-similar approximation

The approach consists in assuming that the flow is locally self-similar. This means that we suppose that $f(\eta, \xi)$ and $f'(\eta, \xi)$ depends slowly in the variable ξ, or that the principal function β varies slowly with respect to ξ. With such an assumption, at dominant order, the self-similar stream function f obeys to the Falkner-Skan equation

$$f''' + ff'' + \beta(1 - f'^2) = 0,$$

Note that such approximation is related to the Blasius series expansion technique [15].

We now compute the boundary condition that must be applied to solve the problem. The velocity field obeys to the usual no slip boundary conditions and we thus have (9a) and (9b) verified. It has been shown in the introduction the necessity of coupling the pressure $P(x)$ with the boundary layer in order to capture quantitatively the physical property of the flow. In order to do so, we embed the fluid from the solid body until a stream line situated at $y = h(x)$. In this domain, the conservation mass law is verified, and using relations (4a) and (4d) :

$$q = \int_0^{h(x)} u(x,y)dy = \int_0^{h(x)} U\delta f'(\eta)\frac{dy}{d\eta}dy = \sqrt{\frac{2\xi}{Re}}\int_0^L f'(\eta)d\eta = \sqrt{\frac{2\xi}{Re}}f(L),$$

Hence we deduce a third boundary condition $f(L) = Q$, with

$$L = \sqrt{\tfrac{Re}{2\xi}}Uh(x)$$
$$Q = \sqrt{\tfrac{Re}{2\xi}}q$$

The stream line at $y = h(x)$ is located at $\eta = L$ in the plane (ξ, η). It is supposed that no tangential stress is applied to this typical stream line. If this latter depends slowly with the coordinate x, then $u_y|_{y=h(x)} = 0$, that corresponds in the (ξ, η) plane to the condition $f''(L) = 0$.

Hence the general problem we need to solve is a boundary value problem, with system (8), and boundary conditions:

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 0 \\ f(L) &= Q \\ f''(L) &= 0 \end{aligned} \qquad (10)$$

We impose four boundary conditions for an ordinary diferential equation that is of order three, as consequence the problem is over-determined and there is no solution in general. The parameter β is choosen as free parameter whose value is adapted in order to satisfy all the boundary conditions (10). Hence, giving a given set (ξ, U) we compute L and Q, then the falkner-Skan equation is solved with the boundary conditions (10) and it is possible to construct a function $\beta(L, Q)$. Properties of this function will be discussed elsewhere [16].

The interest lying on our reduction is that the function $\beta(L, Q)$ may be computed before resolving the complete system. Once this task is performed, the Prandtl equation is approximated with an ordinary differencial equation:

$$\xi_x = U$$
$$U_x = \beta(L, Q)\tfrac{U^2}{2\xi}$$

or equivalently

$$U_\xi = \beta(L, Q)\frac{U}{2\xi}$$

An another nice property captured by this approach is the ability to solve the boundary layer equation in the limit where the velocity $U(x)$ does not obey to the Euler equation, that occurs when the *viscous* fluid invaded all the depth of the fluid. The horizontal velocity is (4d):

$$u = Uf'(\eta),$$

since we did not imposed that $f'(L) = 1$, this means that: if $f'(L) = 1$, U is a potential velocity, whereas in the contrary, U does not represent a potential velocity far away fom the solid body surface. In such a case, this means that the Bernouilli relation is not valid, hence, the boundary layer invaded all the flow, and the fluid becomed viscous. When L tends to infinity, the conditions $f''(L) = 0$ and $f'(L) = 1$ may be verified together [16]. Such situation arise when the boundary layer thickness $\delta(x)$ is small compared to $h(x)$. Hence the aproximation permits the study of variable flow, for which the transition from a flow with an almost inexistant boundary layer to the asymptotic flow, obtained through the lubrification approximation.

4 The Hagen-Poiseuille flow

We shall now consider the cases of flow in the inlet length of a straight channel with flat parallel walls. The horizontal direction is chosen to be parallel to theses planes. At the entrance of the channel, the horizontal velocity is supposed to be high compared to the vertical one. Again, in this example, the boundary layer thickness must be small compared to the distance separating the two planes. There is therefore two fluids: one, in the vicinity of the planes that is *viscous*, and an another one, that can be supposed to be *inviscid* far away from the boundaries. The boundary layer thickness is growing as the measure point is moving far away from the entrance. Further on, it reaches the center of the channel, and the boundary layer has invaded all the domain: the fluid may be considered as *viscous* in this region. Such a transition may be typically analysed using the BLT, and the fluid is described by the Prandtl equation:

$$uu_x + vu_y = -P_x + Re^{-1}u_{yy} \qquad (11)$$

$$u_x + v_y = 0, \qquad (12)$$

The boundary conditions are

$$\begin{aligned} u|_{y=0,2} &= 0 \\ v|_{y=0,2} &= 0 \\ u|_{x=0} &= U_0, \end{aligned} \qquad (13)$$

where the (adimensionalised) distance between the two planes plane is 2. We study solutions that are symmetric with respect to reflection with the $y = 1$

axe : the conditions are transformed into

$$\begin{aligned} u|_{y=0} &= 0 \\ v|_{y=0} &= 0 \\ \partial_y u|_{y=1} &= 0 \\ u|_{x=0} &= U_0 \end{aligned} \tag{14}$$

In the limit of small Reynold number, u is indeed small and the lubrification approximation usefull [14] since Eqs. (12) reduce to the linear equations:

$$\begin{aligned} -P_x + Re^{-1} u_{yy} &= 0 \\ u_x + v_y &= 0 \end{aligned}$$

If the pressure gradient is a constant, then the horizontal velocity has a parabolic profile, and the vertical velocity is zero [14].

$$\begin{aligned} u &= Re P_x \left(\tfrac{y^2}{2} - y\right) \\ v &= 0 \end{aligned}$$

Since the mass flux is conserved, we express the solution as function of $q = \int_0^1 u dy$

$$u = 3q\left(y - \frac{y^2}{2}\right),$$

that is known as the Hagen-Poiseuille profile.

4.1 Approximation with the locally self-similar solutions

In this sub-section, we apply our approximation described in section 3. We now use the Görtler tranformation, and reduce the problem (12) to the Görtler equation (5), and suppose that the (self-similar) *stream function* $f(\xi, \eta)$ just depends in the variable η.

$$f''' + ff'' + \beta(1 - f'^2) = 0. \tag{15}$$

Due to the symmetry of the problem, we fix the boundary stream line $y = h(x)$ in the middle of the channel $y = 1$. The boundary conditions for this equation are

$$f(0) = f'(0) = 0 \qquad f''(L) = 0 \qquad F(L) = Q \tag{16a}$$

$$L = \sqrt{\frac{Re}{2\xi}}U \tag{16b}$$

$$Q = \sqrt{\frac{Re}{2\xi}}q \tag{16c}$$

and the the equation for the U field

$$U_\xi = \beta(L, Q)\frac{U}{2\xi}$$

4.1.1 Self similar solution a the entrance of the pipe

At the entrance of the pipe ($x = 0$), the longitudinal velocity is constant between the two planes, and the size of the boundary layer δ_0, is zero. In this section we compute the behavior of the fluid for $x \ll 1$. Inside the boundary layer, the horizontal velocity is small compared to those at $y = 1$, and by mass conservation we deduce that, at first order

$$q = \int_0^1 u dy \simeq (1 - \delta_0)U,$$

where $\delta_0 = \alpha\delta \ll 1$, the parameter α is a geometrical factor that may will be computed later; hence the velocity is $U = q(1 + \delta_0)$. The longitudinal coordinate ξ is obtained at dominant order using (4a):

$$\xi = qx; \tag{17}$$

this implies that as expected δ tends to zero in this limit, and as consequence the parameter L diverges at the entrance point. We now compute the principal function β:

$$\beta = 2\xi\frac{U_\xi}{U} = 2\xi\frac{\alpha\delta_\xi}{1 + \alpha\delta}$$

Using the definition for δ given in (4c), we deduce $\delta_\xi = \delta/(2\xi)$ and

$$\beta = \alpha\delta\,(1 - \alpha\delta) \tag{18}$$

Since δ tends to zero at the entrance of the pipe, β tends also to zero, and the entrant profile is of Blasius type. The geometrical factor α may be now directly computed. With the definitions (16b) and (16c) we can write

$$L - Q = \frac{1}{\delta}(U - q)$$

For large values of L, it can be shown that $\beta = 0$ implies $L - Q = 1.217$ [16]. By comparison with the definition of α, we thus predict that $\alpha = 1.217/q$. We then conclude that the horizontal velocity in the vicinity of the entrance and at $y = 1$ is written

$$U = q\left(1 + \alpha\sqrt{\frac{2qx}{Re}}\right) = q\left(1 + 1.72\sqrt{\frac{x}{qRe}}\right) \tag{19}$$

The Blasius serie expansion (see ref. [15]) predicts the result (19) but with a prefactor 1.73 instead of 1.72. Using the Bernouilli relation, we deduce the behavior of the pressure gradient P_x near the entrance point:

$$P_x = -1.217\frac{q^{3/2}}{\sqrt{xRe}} \tag{20}$$

We integrated numerically the boundary layer equation (12) (using a Cranck Nickolson scheme) with a constant initial profile for u, this divergence is clearly observed in Fig. 2.

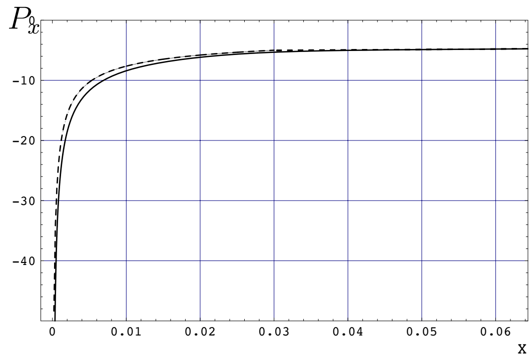

Fig. 2. Comparison of the pressure gradients obtained by the locally self similar approximation (dashed line), and by direct numerical integration of the Eqs. (12) (solid line).

4.1.2 Asymptotic Hagen-Poiseuille self similar solution

We are now interested in the profiles far away from the channel entrance, i.e. $1 \ll x$. It is expected that in this channel region, the fluid is completely viscous, and the velocity profiles should be of Hagen-Poiseuille type. Hence, let us suppose that $P_x = -\Gamma$ is constant. Since $P_x = -UU_x$, we deduce

$$U = \sqrt{2\Gamma\left(x - x_1\right)} \simeq \sqrt{2\Gamma x} \tag{21}$$

It is possible to compute the behavior for variable ξ using the definition (4a):

$$\xi = \frac{2x}{3}\sqrt{2\Gamma x} \tag{22}$$

We can evaluate the principal function β:

$$\beta = 2\xi\frac{U_\xi}{U} = \frac{2}{3} \tag{23}$$

It remains to study the asymptotic profile of the velocity: we need to solve the Falkner Skan equation with $\beta = \frac{2}{3}$. For this value of β, no analytic solution is known. Nevertheless, it is possible to find the profile since the variable L tends to zero, as x tends to infinity:

$$L = \sqrt{\frac{Re}{2\xi}}U = \sqrt{\frac{3Re}{4}}\left(\frac{2\Gamma}{x}\right)^{\frac{1}{4}} \tag{24}$$

Since $0 < \eta < L$, the Falkner skan equation is then solved using a Taylor serie expansion for f, at $\eta = 0$:

$$f(\eta) = f(0) + f'(0)\eta + \frac{1}{2}f''(0)\eta^2 + \frac{1}{6}f'''(0)\eta^3$$

Due to the boundary conditions (16a) at $\eta = 0$ we then have

$$f(\eta) = \frac{1}{2}f''(0)\eta^2 + \frac{1}{6}f'''(0)\eta^3$$

By using the Falkner-Skan equation we conclude that $f'''(0) = -\beta = -\frac{2}{3}$. The no tangential stress at $y = 1$ implies that $f''(L) = 0$, and we deduce the stream function f:

$$f(\eta) = \frac{1}{9}\left(3\eta^2 L - \eta^3\right)$$

Note that $f'(L) = L^3/3$, and as expected, U does not represent a potential velocity: the fluid is viscous in this part of the channel. Since $u = Uf'(\eta)$, we deduce finally that

$$u = \frac{1}{3}U\eta(2L - \eta) = Re\Gamma\left(y - \frac{y^2}{2}\right) \tag{25}$$

It remains to apply the mass conservation law $\int_0^1 udy = q$ and we conclude that

$$\Gamma = \frac{3q}{Re} \tag{26}$$

$$u = 3q\left(y - \frac{y^2}{2}\right), \tag{27}$$

and this profile corresponds to the Hagen-Poiseuille solution. This solution is the asymptotic state of the direct simulation of the Prandtl equation (12) and the model presenteds here, since the pressure gradient converges to a constant (see Fig. 2) and the profile of the horizontal velocity becomes parabolically shaped (see Fig. 3).

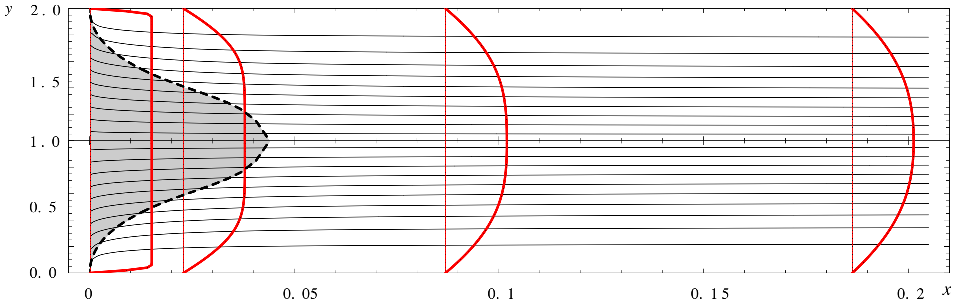

Fig. 3. Numerical result for the integration of the Falkner-Skan approximation. Stream-lines (solid black line) and some velocity profile (solid gray lines) of the Poiseuille flow. The dashed curve defines the height of the boundary layer. The region where the fluid is *inviscid* is filled in light gray.

4.2 Comments about the locally self similar approximation

The basis of the approximation is to suppose that the profile of the velocity is locally self similar, but with the principal function β slowly varying in ξ or x. In the Poiseuille geometry, at the entrance of the pipe, β is equal to zero, and tends to $\frac{2}{3}$ as x tends to infinity. Thus, there are not large variations of the principal function in the longitudinal direction. Our approach permits to simplify the Prandtl equation to a set of ordinary differential equations, and the agreement between the simplified model and the complete system is quite good. The Görtler transformation is general and may be applied to quite different geometries, thus our method for approximating the Prandtl equation has the advantage to be used for not academic problem and we have applied this method for describing the radial hydraulic jump [17] with success [16].

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